

GROUP ANALYSIS OF NONLINEAR INTERNAL WAVES IN OCEANS III: Additional conservation laws¹

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Abstract. Using the maximal Lie algebra of point symmetries of a system of nonlinear equations used in geophysical fluid dynamics, two conservation laws are found in addition to the conservation of energy.

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1 Introduction

The maximal group of Lie point symmetries of the system

$$\Delta\psi_t - g\rho_x - fv_z = \psi_x\Delta\psi_z - \psi_z\Delta\psi_x, \quad (1.1)$$

$$v_t + f\psi_z = \psi_x v_z - \psi_z v_x, \quad (1.2)$$

$$\rho_t + \frac{N^2}{g}\psi_x = \psi_x\rho_z - \psi_z\rho_x \quad (1.3)$$

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has been presented in [1]. It is generated by the infinite-dimensional Lie algebra spanned by the following operators:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = a(t) \frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t}, \\
X_5 &= b(t) \left[\frac{\partial}{\partial x} - f \frac{\partial}{\partial v} \right] + b'(t) z \frac{\partial}{\partial \psi}, \\
X_6 &= c(t) \left[\frac{\partial}{\partial z} + \frac{N^2}{g} \frac{\partial}{\partial \rho} \right] - c'(t) x \frac{\partial}{\partial \psi}, \\
X_7 &= x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}, \\
X_8 &= t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3\psi \frac{\partial}{\partial \psi} - 2fx \frac{\partial}{\partial v} + 2 \frac{N^2}{g} z \frac{\partial}{\partial \rho}, \\
X_9 &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} - \frac{1}{f} [g\rho + (f^2 - N^2)z] \frac{\partial}{\partial v} + \frac{1}{g} [fv + (f^2 - N^2)x] \frac{\partial}{\partial \rho}.
\end{aligned} \tag{1.4}$$

2 Conservation law provided by semi-dilation

Consider the operator X_8 from (1.4),

$$X_8 = t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3\psi \frac{\partial}{\partial \psi} - 2fx \frac{\partial}{\partial v} + 2 \frac{N^2}{g} z \frac{\partial}{\partial \rho} \tag{2.1}$$

It generates the following one-parameter transformation group with the parameter ε :

$$\begin{aligned}
\bar{t} &= te^\varepsilon, \quad \bar{x} = xe^{2\varepsilon}, \quad \bar{z} = ze^{2\varepsilon}, \quad \bar{\psi} = \psi e^{3\varepsilon}, \\
\bar{v} &= v + fx(1 - e^{2\varepsilon}), \quad \bar{\rho} = \rho - \frac{N^2}{g}(1 - e^{2\varepsilon}).
\end{aligned} \tag{2.2}$$

Since some of variables, namely t, x, z and ψ are subjected to dilations while two other variables transform otherwise, we call (2.2) the *semi-dilation group*. Let us construct the conserved vector provided by this group.

2.1 Computation of the conservation law density

We will use the following formula for computing the density of conservation laws (see Eq. (3.23) in [2])

$$C^1 = -v W^1 - \frac{g^2}{N^2} \rho W^2 - \psi_x D_x(W^3) - \psi_z D_z(W^3), \tag{2.3}$$

where (see Eq. (3.16) in [2])

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, 2, 3. \quad (2.4)$$

These formulas are written using the notation t, x, z, v, ρ, ψ .

In the case of the operator (2.1) the quantities (2.4) are written:

$$\begin{aligned} W^1 &= -2fx - tv_t - 2xv_x - 2zv_z, \\ W^2 &= 2\frac{N^2}{g}z - t\rho_t - 2x\rho_x - 2z\rho_z, \\ W^3 &= 3\psi - t\psi_t - 2x\psi_x - 2z\psi_z. \end{aligned} \quad (2.5)$$

Substituting (2.5) in (2.4) we obtain upon simple calculations:

$$\begin{aligned} C^1 &= 2(fxv - gz\rho) - |\nabla\psi|^2 + (xD_x + zD_z)\left(v^2 + \frac{g^2}{N^2}\rho^2\right) \\ &+ (xD_x + zD_z)(|\nabla\psi|^2) + t\left[vv_t + \frac{g^2}{N^2}\rho\rho_t + \psi_x\psi_{xt} + \psi_z\psi_{zt}\right]. \end{aligned} \quad (2.6)$$

We can drop the last term in (2.6) because it can be written in the divergent form upon elimination of v_t, ρ_t and ψ_t by using Eqs. (1.1)-(1.3). Indeed, it is shown in [2], Section 4.6, that the expression in the square brackets (cf. Eq. (4.11) in [2]) evaluated on the solutions of Eqs. (1.1)-(1.3) has the divergent form. Multiplication by t does not violate this property. Then we use the identities

$$\begin{aligned} (xD_x + zD_z)\left(v^2 + \frac{g^2}{N^2}\rho^2\right) &= -2\left(v^2 + \frac{g^2}{N^2}\rho^2\right) \\ &+ D_x\left[x\left(v^2 + \frac{g^2}{N^2}\rho^2\right)\right] + D_z\left[z\left(v^2 + \frac{g^2}{N^2}\rho^2\right)\right], \\ (xD_x + zD_z)(|\nabla\psi|^2) &= -2\psi^2 + D_x[x(|\nabla\psi|^2)] + D_z[z(|\nabla\psi|^2)], \end{aligned}$$

drop the divergent type terms and obtain the following conserved density:

$$C^1 = 2\left(fxv - gz\rho - \frac{1}{2}|\nabla\psi|^2\right) - 2\left(v^2 + \frac{g^2}{N^2}\rho^2 + |\nabla\psi|^2\right). \quad (2.7)$$

Finally we note that the last term in (2.7) is the energy density (see Eq. (4.17) in [2]). Therefore we eliminate it and conclude that the invariance under the semi-dilation with the generator (2.1) provides the conservation law with the density

$$P = fxv - gz\rho - \frac{1}{2}|\nabla\psi|^2. \quad (2.8)$$

2.2 Conserved vector

Let us find the components C^2 , C^3 of the conserved vector with the density (2.8). We will apply the procedure used in [2], Section 4.7. We have:

$$D_t(P) = fxv_t - gz\rho_t - (\psi_x\psi_{xt} + \psi_z\psi_{zt}).$$

Using Eqs. (1.1)-(1.3), we obtain:

$$\begin{aligned} D_t(P) \Big|_{(1.1)-(1.3)} &= -f^2x\psi_z + fx\psi_xv_z - fx\psi_zv_x + N^2z\psi_x \\ &\quad - gz\psi_x\rho_z + gz\psi_z\rho_x - D_x(\psi\psi_{xt}) - D_z(\psi\psi_{zt}) + \psi\Delta\psi_t. \end{aligned}$$

One can rewrite this equation, using Eq. (4.23) from [2], in the following form:

$$\begin{aligned} D_t(P) \Big|_{(1.1)-(1.3)} &= D_x\left(N^2z\psi + fx\psi v_z - gz\psi\rho_z - \psi\psi_{xt} + \frac{1}{2}\psi^2\Delta\psi_z\right) \\ &\quad - D_z\left(f^2x\psi + fx\psi v_x - gz\psi\rho_x + \psi\psi_{zt} + \frac{1}{2}\psi^2\Delta\psi_x\right). \end{aligned}$$

Thus, the generator (2.1) provides the conservation law

$$D_t(P) + D_x(C^2) + D_z(C^3) = 0$$

with the density P given by (2.8) and the flux given by the equations

$$\begin{aligned} C^2 &= -N^2z\psi - fx\psi v_z + gz\psi\rho_z + \psi\psi_{xt} - \frac{1}{2}\psi^2\Delta\psi_z, \\ C^3 &= f^2x\psi + fx\psi v_x - gz\psi\rho_x + \psi\psi_{zt} + \frac{1}{2}\psi^2\Delta\psi_x. \end{aligned}$$

2.3 Conserved density P of the generalized invariant solution

If we substitute in (2.8) the generalized invariant solution (5.28)-(5.30) from our paper [2],

$$\begin{aligned} \psi &= A(\lambda) \cos(\omega t) + B(\lambda) \sin(\omega t), \\ v &= \frac{fm}{\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)], \\ \rho &= \frac{kN^2}{g\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)], \end{aligned}$$

we obtain:

$$P = \frac{1}{\omega} (f^2 m x - N^2 k z) [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)] \\ - \frac{k^2 + m^2}{2} [A'(\lambda) \cos(\omega t) + B'(\lambda) \sin(\omega t)]^2.$$

3 Conservation law provided by the rotation

Taking the rotation generator X_9 from (1.4) and proceedings as in Section 2 we obtain the following conserved density:

$$Q = v\rho + fx\rho - \frac{N^2}{g} zv. \quad (3.1)$$

Writing Eqs. (1.1)-(1.3) by using the Jacobians $J(\psi, v) = \psi_x v_z - \psi_z v_x$, etc., we have:

$$D_t(Q) \Big|_{(1.1)-(1.3)} = v \left[J(\psi, \rho) - \frac{N^2}{g} \psi_x \right] + \rho [J(\psi, v) - f\psi_z] \\ + fx \left[J(\psi, \rho) - \frac{N^2}{g} \psi_x \right] - \frac{N^2}{g} z [J(\psi, v) - f\psi_z]. \quad (3.2)$$

The reckoning shows that

$$vJ(\psi, \rho) + \rho J(\psi, v) = D_z(v\rho\psi_x) - D_x(v\rho\psi_z), \\ xJ(\psi, \rho) - \rho\psi_z = D_z(x\rho\psi_x) - D_x(x\rho\psi_z), \\ zJ(\psi, v) + v\psi_x = D_x(zv\psi_z) - D_z(zv\psi_x), \\ z\psi_z - x\psi_x = D_z(z\psi) - D_x(x\psi).$$

Substituting these expressions in Eq. (3.2) we conclude that the rotation generator X_9 provides the conservation law

$$D_t(Q) + D_x(C^2) + D_z(C^3) = 0$$

with the density P given by (3.1) and the flux given by the equations

$$C^2 = \left[v\rho + fx\rho - \frac{N^2}{g} zv \right] \psi_z + \frac{N^2}{g} fz\psi, \\ C^3 = \left[\frac{N^2}{g} zv - v\rho - fx\rho \right] \psi_x - \frac{N^2}{g} fx\psi.$$

4 Summary of conservation laws

It has been demonstrated in [2] that the system of nonlinear equations (1.1)-(1.3) is self-adjoint. This property of the system has been used for deriving local conservation laws applying the method developed in [3] to the infinitesimal symmetries (1.4). Some of the conservation laws associated with these symmetries are *trivial*, i.e. have vanishing densities. But five conservation laws are nontrivial.

The nontrivial conservation laws obtained in [2] and in the present paper are summarized below. For the convenience of the reader, we formulate them both in the integral and differential forms.

4.1 Conservation laws in integral form

$$\frac{d}{dt} \iint v \, dx dz = 0. \quad (4.1)$$

$$\frac{d}{dt} \iint \rho \, dx dz = 0. \quad (4.2)$$

$$\frac{d}{dt} \iint \left[v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right] dx dz = 0. \quad (4.3)$$

$$\frac{d}{dt} \iint \left[f x v - g z \rho - \frac{1}{2} |\nabla \psi|^2 \right] dx dz = 0. \quad (4.4)$$

$$\frac{d}{dt} \iint \left[v \rho + f x \rho - \frac{N^2}{g} z v \right] dx dz = 0. \quad (4.5)$$

4.2 Conservation laws in differential form

$$D_t(v) + D_x(v\psi_z) + D_z(f\psi - v\psi_x) = 0. \quad (4.1')$$

$$D_t(\rho) + D_x\left(\frac{N^2}{g}\psi + \rho\psi_z\right) + D_z(-\rho\psi_x) = 0. \quad (4.2')$$

$$\begin{aligned}
& D_t \left(v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right) \\
& + D_x \left(2g\rho\psi + v^2\psi_z + \frac{g^2}{N^2} \rho^2\psi_z - 2\psi\psi_{xt} + \psi^2\Delta\psi_z \right) \\
& + D_z \left(2fv\psi - v^2\psi_x - \frac{g^2}{N^2} \rho^2\psi_x - 2\psi\psi_{zt} - \psi^2\Delta\psi_x \right) = 0.
\end{aligned} \tag{4.3'}$$

$$\begin{aligned}
& D_t \left(f xv - gz\rho - \frac{1}{2} |\nabla \psi|^2 \right) \\
& + D_x \left(-N^2 z\psi - fx\psi v_z + gz\psi\rho_z + \psi\psi_{xt} - \frac{1}{2} \psi^2 \Delta\psi_z \right) \\
& + D_z \left(f^2 x\psi + fx\psi v_x - gz\psi\rho_x + \psi\psi_{zt} + \frac{1}{2} \psi^2 \Delta\psi_x \right) = 0.
\end{aligned} \tag{4.4'}$$

$$\begin{aligned}
& D_t \left(v\rho + fx\rho - \frac{N^2}{g} zv \right) \\
& + D_x \left(\left[v\rho + fx\rho - \frac{N^2}{g} zv \right] \psi_z + \frac{N^2}{g} fz\psi \right) \\
& + D_z \left(\left[\frac{N^2}{g} zv - v\rho - fx\rho \right] \psi_x - \frac{N^2}{g} fx\psi \right) = 0.
\end{aligned} \tag{4.5'}$$

The conservation law (4.3) defines the energy of the system. It seems that the conservation laws (4.4) and (4.5), unlike (4.3), do not have direct analogies in mechanics and should be investigated from point of view of their physical significance.

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